Optimal replacement and overhaul decisions with imperfect maintenance and warranty contracts

R. Pascual
Department of Mechanical Engineering, Universidad de Chile, Casilla 2777, Santiago, Chile
Phone: +56-2-6784591 Fax:+56-2-6896057
rpascual@ing.uchile.cl

Abstract

In this article we develop a model to help a maintenance decision-making situation of a given equipment. We propose a novel model to determine optimal life-cycle duration and intervals between overhauls by minimizing global maintenance costs. We consider a situation where the customer, which owns the equipment, may negotiate a better warranty contract by offering an improved preventive maintenance program for the equipment. The equipment receives three kind of actions: repairs, overhauls, and replacement. An overhaul represents an imperfect maintenance action, that is, the failure rate is improved but not a point that the equipment is as good as new. Corrective maintenance actions are minimal, in the sense that the failure rate after each repair is the same as before the failure. The proposed strategy surpasses others seen in the literature since it considers at the same time the warranty negotiation situation and the optimal life-cycle duration under imperfect preventive actions. We also propose a simplified approach that facilitates the task of implementing the method in standard spreadsheet solvers.

Keywords: imperfect maintenance, warranty, management, life-cycle cost.

1. Introduction

Many systems are sold with a warranty that offers protection to the buyers against early failures during the infancy of the equipment and as a medium of promotion to the vendor. When the warranty period tends to be large, degradation also appears during it and in this case preventive maintenance plays an important role in order to reduce the failure rate, and its evolution in time.

Offering periods of warranty implies extra costs for the vendor. There are repair costs (corrective maintenance) and possibly penalty costs to be paid, which are associated to downtime. A preventive maintenance program may reduce such corrective costs. Given that the buyer does not pay repairs during the warranty period, there are no incentives for him to spend in preventive actions (in the case that the warranty also covers for the downtime costs). For the vendor, it is convenient to perform preventive actions only if they cost less than the expected corrective costs.

From the buyers’ point of view, investing in preventive maintenance during and after the warranty period may have significant effects on the life-cycle costs. Consequently, it may be convenient to define preventive policies all along the life-cycle. The aim of this work is to present a general framework for this situation from the point of view of the buyer.

We consider three kind of maintenance actions: minimal repair (as good as before the failure), imperfect overhaul (between as good as after the previous overhaul and as good as before the overhaul) or replaced (as good as new). Each action has its own costs and may depend on variables such as age and/or quality. We will consider a single component analysis, neglecting scale economies that may appear from negotiating for multi-component systems. A cost analysis for systems in series, in parallel and a combination of both may be found in Bai and Pham[2].

Modelling of the system improvement due to imperfect maintenance is crucial to establish the cost model to be optimized. Malik[9] introduced the concept of virtual age, which essentially says that the system is younger than before the action by some interval \( T_x \). A limitation of this model is that it does not alter the reference failure rate function. Nakagawa[11] assumed that the action is minimal with probability \( a \) and perfect with probability \( (1 - a) \). As it is referred to minimal and perfect actions, this model implies that as time passes the quality of the overhauls must be improved in order to keep \( a \) constant. This has some consequences: if the original failure rate without overhauls of the system is a power function of time, the failure rate is always bounded. Zhang and Jardine[7] propose a failure rate model where after an overhaul, it is between as good as before and as good as after previous overhaul. This model does not bound the failure rate as Nakagawa’s but due to the discontinuities in the failure rate function, it may be difficult to evaluate the number of expected failures during the warranty period, which is variable in our model. Djamaludin et al.[4] propose to use a continuous failure rate function whose growth parameter is dependent on the quality (cost) of the maintenance policy. For maximum quality the failure rate is constant, for minimum quality, the failure rate corresponds to the original failure rate when no overhauls are performed. A serious limitation of this approach is how to model the relationship between the quality of the preventive policy and the failure rate. A summary of research on imperfect maintenance is offered in Pham and Wang[5].

Considering warranty, Djamaludin et al.[4] develop a framework to study preventive policies when the vendor offers a given period of warranty where he pays labor, materials and downtime costs if a failure occurs. Under this premise, the buyer is not necessarily committed to preventive maintenance. Jack and Dagpunar[6] use a virtual age model to determine the quality and the period between overhauls. In their model, the optimal solution is complete renewal at each overhaul (age 0). Jung and Park[8] study the optimal periodic preventive policies following the expiration of warranty. They
use the expected maintenance cost rate per unit time from the buyer’s perspective. They also use a virtual age model for the failure rate and consider that preventive activities start just after the end of the warranty. This limits the optimality of the preventive maintenance since early preventive actions may reduce even further the life-cycle costs\cite{4}. Research dealing on warranty cost analysis has been summarized in Blischke and Murthy\cite{3} and Murthy and Djamaludin\cite{10}.

The model presented here considers that preventive policies are taken from the beginning of the life-cycle. In this way, the failure rate is reduced and costs associated to corrective maintenance are reduced.

We shall propose a failure rate model that takes the advantages of both the model of Zhang and Jardine and the one by Djamaludin et al.. Based on the proposed model we minimize the expected cost per unit-time. It allows an optimal decision on life-cycle duration and the number of periodic overhauls to perform during it. It also permits a negotiation of the warranty period with the vendor. We show results for the case when the reference failure rate function is growing exponentially with time. We illustrate the methodology through a numerical example and we obtain optimal values for the number of overhauls and the life-cycle duration.

The improvements made over the reference models are:

- life-cycle duration is a decision variable;
- we obtain an optimal value for the warranty period to be negotiated with the vendor;
- we consider explicitly the downtime costs which are not paid in general by the vendors during the warranty period;
- we consider the cost of an overhaul as a function of its quality;
- we propose a continuous failure-rate model which is equivalent to the model of Zhang and Jardine but it is easier to evaluate at any instant.

2. General model

2.1. Statement of the problem

Equipment breaks down from time to time, requiring repairs. Also, while the equipment is being repaired, there is a loss in production output. In order to reduce the number of failures, we can periodically overhaul the equipment and perform preventive actions. After some time it may be economically convenient to replace the equipment by another new one.

Our purpose is to determine the optimal life-cycle duration and the interval between overhauls that minimize the global cost per unit time.

2.2. Construction of model

Let us consider the following conditions:

- Equipment receives \(n - 1\) overhauls during its life; \(n \geq 1\), integer,positive
- the interval between overhauls \(T_i\) is constant. The life-cycle duration \(T_l\) is then \(T_l = nT_i\) \(2\)
- an overhaul improves the equipment in term of its failure rate \(\lambda\);
- all repairs are minimal, that is, they only return the equipment to production but they don’t improve the failure rate.
- The quality of an overhaul (as well as its cost) is dependent on the improvement factor \(p\):
  \[0 < p < 1\] \(3\)
- Material and tradesmen to perform a repair cost \(c_{im}\);
- downtime cost of a repair is \(c_{fm}\);
- overall cost of an overhaul is \(c_o\) (it includes: material, tradesmen downtime costs);
- overall cost of a replacement is \(c_r\) (investment, labor, material, downtime costs);
- failure rate with periodic overhauls is \(\lambda(t)\);
- failure rate if no overhauls are performed is \(\tilde{\lambda}(t)\);
- The vendor only pays labour and material costs during the warranty period \(T_w\); downtime costs are assumed by the customer;
- The customer and the vendor agree to negotiate an extended warranty period if the customer performs overhauls during the duration of the contract; we have:
  \[T_w < T_l\] \(4\)
- repairs beyond the warranty period are paid by the customer;
- recovery value of the equipment is negligible;
- the quality \(p\) is considered constant, as well as its cost,
- the mean time to repair is negligible in front of the mean time between failures;
- the vendor offers a baseline warranty period \(\bar{T}_w\), where the customer is not obliged to perform overhauls; the customer may negotiate an extension to the warranty so,
  \[\bar{T}_w \leq T_w\] \(5\)

Our aim is to determine the number of overhauls, their interval \(T_i\) (or equivalently the life-cycle duration \(T_l\)), and the warranty interval \(T_w\), that minimize the total expected cost per unit time \(c_g\).
2.3. Cost model
The expected number of failures during the warranty period is given by:
\[ \int_0^{T_w} \lambda(t)\,dt \]
and during the rest of the life-cycle by
\[ \int_{T_w}^{T_i} \lambda(t)\,dt \]
The total life-cycle cost for the customer is given by
\[ C_g(n, T_i, p, T_w) = c_r + c_o(p)(n-1) + c_{fm} \int_0^{T_w} \lambda(t)\,dt + c_{im} \int_{T_w}^{T_i} \lambda(t)\,dt \]
and for the vendor by
\[ c_{im} \int_0^{T_w} \lambda(t)\,dt \]
the life-cycle cost per unit time for the customer is then
\[ c_g(n, T_i, p, T_w) = \frac{C_g(n, T_i, p, T_w)}{T_i} \]

2.4. Constraints
If no negotiation occurs, the vendor agrees to pay the labor and material required to minimally repair the equipment during the warranty period:
\[ c_{im} \int_0^{T_w} \lambda(t)\,dt \]
On the other hand, the customer performs overhauls during the life of the equipment in order to reduce the number of failures during the life of the equipment. This would reduce the expected number of failures during the warranty period and the expected cost for the vendor. Performing overhauls imply costs to be paid by the customer so the vendor could compensate his efforts by extending the warranty period. The vendor does not want to increase his expected cost so he would agree to extend the warranty if the expected cost during the extended warranty does not surpass the expected cost for the baseline warranty, that is:
\[ c_{im} \int_0^{T_w} \lambda(t)\,dt \leq c_{im} \int_0^{T_w} \lambda(t)\,dt \]
or, in terms of number of failures,
\[ \int_0^{T_w} \lambda(t)\,dt \leq \int_0^{T_w} \lambda(t)\,dt \]
Given that \( \lambda(t) \) is non negative, we observe that the last term in (8):
\[ c_{im} \int_{T_w}^{T_i} \lambda(t)\,dt \]
is decreasing as \( T_w \) increases. The minimization of the total cost of the customer implies forcing the constraint (9) to the equality:
\[ \int_0^{T_w} \lambda(t)\,dt = \int_0^{T_w} \lambda(t)\,dt \]
as a consequence, \( T_w \) is not an active decision variable in the sense that it may be obtained once \( \lambda(t) \) is modelled. Of course, the vendor will agree to extend the warranty only if at least one overhaul is performed during \([0, T_w] \):
\[ T_i \leq T_w \]

2.5. Discontinuous failure rate model
The failure rate is a crucial indicator of the equipment condition since it permits failure forecasting and establish appropriate preventive measures like overhauls. We will consider that the failure rate after an overhaul fails between as bad as just before and as well as just after the previous overhaul with some improvement factor \( p \in (0, 1) \).
Let \( \lambda_k(t) \) be the failure rate after the \( k-th \) overhaul. We will express the failure rate as:
\[ \lambda_k(t) = p\lambda_{k-1}(t-T_i) + (1-p)\lambda_{k-1}(t) \]
which produce discontinuities in \( \lambda(t) \) as observed in figure 1.

Figure 1: Model from Zhang and Jardine vs proposed (from the example, \( p = 0.7 \))
In a general situation the aging process shows the pattern observed in figure 1. If the improvement factor \( p \) is 0, then
\[ \lambda_k(t) = \lambda_{k-1}(t) \]
in other words, the failure rate is the same as before the overhaul so it may be considered as a minimal repair (see figure 2).

Figure 2: Perfect \((p = 1)\) and minimal \((p = 0)\) overhaul
If the improvement factor $p$ is 1, 

$$\lambda_k(t) = \lambda_{k-1}(t-T)$$

the overhaul returns the failure rate to the level obtained just after the previous overhaul. Since all overhauls have the same level and their periodicity is constant, overhaul may be considered as a replacement (figure 2).

It may be proven that

$$\int_0^{T_i} \lambda_i(t)dt = \sum_{i=0}^n \left( \frac{n}{i} \right) p^{n-i}(1-p)^{i-1} \int_0^{T_i} \hat{\lambda}(t)dt$$  \hspace{1cm} (12)

### 2.6. Exponential growth model

If the failure rate with no overhaul follows

$$\hat{\lambda}(t) = e^{\alpha_0 + \alpha_1 t}, \alpha_1 > 0$$

from (12) and (13) we have

$$\int_0^{T_i} \lambda_i(t)dt = \sum_{i=0}^n \left( \frac{n}{i} \right) p^{n-i}(1-p)^{i-1} \int_0^{\hat{\lambda}(t)} dt$$

$$= \sum_{i=0}^n \left( \frac{n}{i} \right) p^{n-i}(1-p)^{i-1} e^{\alpha_0 \left( e^{\alpha_1 T} - 1 \right)}$$

$$= \frac{e^{\alpha_0}}{(1-p)\alpha_1} \sum_{i=0}^n \left( \frac{n}{i} \right) p^{n-i} \left[ (1-p) \left( e^{\alpha_1 T} \right) \right]^{i-1}$$

$$= \frac{e^{\alpha_0}}{(1-p)\alpha_1} \left[ p - (1-p) e^{\alpha_1 T} \right]^{n-1}$$

### 2.7. Cost of an overhaul

It is logical to impose a dependency between the quality of an overhaul and its cost, we propose

$$c_o(p) = c_{o, \text{min}} e^{\beta p}$$  \hspace{1cm} (14)

where

$$\beta = \log \frac{c_{o, \text{min}}}{c_{o, \text{max}}}$$

and $c_{o, \text{min}}$ y $c_{o, \text{max}}$ are minimum and maximum (complete renewal) costs for an overhaul (see figure 3).

![Figure 3: Overall cost of an overhaul as a function of $p$](image)

### 3. Proposed model

The model presented in §2.5 show discontinuities every time an overhaul is performed. Given that the replacement problem considers a long term, instantaneous failure rate values are not important to the optimization problem and a long term approximation is useful to easily determine the number of failures during the life-cycle and during the warranty period.

#### 3.1. Equivalent parameters

By observation of figure (1), let us consider a long term failure rate model should produce the same number of failures for each interval and its cost, we propose

$$\hat{\lambda}(t) = e^{\hat{\alpha}_0 + \hat{\alpha}_1 t}$$  \hspace{1cm} (15)

where $\hat{\alpha}_0$ and $\hat{\alpha}_1$ are determined by $p$. To estimate these parameters we exploit (11),

$$\lambda_1(T_i) = p\lambda_{\alpha}(0) + (1-p)\lambda_0(T_i)$$

If we consider the model presented in §2.6

$$\lambda_1(T_i) = p e^{\alpha_0} + (1-p) e^{\alpha_0 + \alpha_1 T_i}$$  \hspace{1cm} (16)

We may estimate $\hat{\alpha}_1$ by using the points $(0, \alpha_0)$ and $(T, \lambda_1(T))$ which give us a lower bound for the failure rate at any instant,

$$\hat{\lambda}_{\text{inf}}(t) = e^{\alpha_0 + \hat{\alpha}_1 t}$$  \hspace{1cm} (17)

with

$$t \in [0, T_i]$$

substituting (16) into (17),

$$e^{\alpha_0 + \hat{\alpha}_1 T_i} = p e^{\alpha_0} + (1-p) e^{\alpha_0 + \alpha_1 T_i}$$  \hspace{1cm} (18)

and we obtain,

$$\hat{\alpha}_1 = \frac{\log \left( p e^{\alpha_0} + (1-p) e^{\alpha_0 + \alpha_1 T_i} \right) - \alpha_0}{T_i}$$  \hspace{1cm} (19)

In order to estimate $\hat{\alpha}_0$, we consider that the continuous model should produce the same number of failures for each interval $T_i$ and that the slope in a loglinear diagram is $\hat{\alpha}_1$,

$$\int_0^{T_i} \hat{\lambda}(t, \hat{\alpha}_1) dt = \int_0^{T_i} \hat{\lambda}(t) dt$$

$$\frac{e^{\alpha_0}}{\hat{\alpha}_1} \left( e^{\alpha_1 T_i} - 1 \right) = \frac{e^{\alpha_0}}{\hat{\alpha}_1} \left( e^{\alpha_1 T_i} - 1 \right)$$

and we obtain,

$$\hat{\alpha}_0 = \log \left( e^{\alpha_0} \frac{\hat{\alpha}_1}{\alpha_1} \frac{e^{\alpha_1 T_i} - 1}{e^{\alpha_1 T_i} - 1} \right)$$  \hspace{1cm} (20)

The use of (17) greatly simplifies the evaluation of constraint (10) and facilitates the optimization process.
3.2. Optimal value for the warranty period

If the failure rate with no overhauls follows (13), the expected number of failures during the reference warranty period $T_w$ is:

$$
\int_0^{T_w} \lambda(t) \, dt = \frac{e^{\alpha_0}}{\alpha_1} \left( e^{\alpha_1 T_w} - 1 \right)
$$

then, the non-linear constraint (9) takes the form

$$
e^{\hat{\alpha}_0} \left( e^{\hat{\alpha}_1 T_e} - 1 \right) = \frac{e^{\alpha_0}}{\alpha_1} \left( e^{\alpha_1 T_w} - 1 \right)
$$

from where we obtain $T_w$ explicitly:

$$
T_w = \frac{\log \left( \frac{e^{\hat{\alpha}_0} \hat{\alpha}_1 \left( e^{\hat{\alpha}_1 T_e} - 1 \right) + 1}{\hat{\alpha}_1} \right)}{\hat{\alpha}_1} \quad (21)
$$

4. Numerical example

Let us consider the following data, which are very similar to those used in reference [7], considering the extra parameters needed for our model:

$$
c_r = 200 \text{ KUSD/renewal}
$$

$$
c_{o,\text{min}} = 8 \text{ KUSD/overhaul}
$$

$$
c_{o,\text{max}} = 32 \text{ KUSD/overhaul}
$$

$$
c_{fm} = 1 \text{ KUSD/repair}
$$

$$
c_{im} = 1 \text{ KUSD/repair}
$$

$$
\bar{T}_w = 730 \text{ days}
$$

the reference failure rate (with no overhaul) follows

$$
\bar{\lambda}(t) = e^{-15+0.01t}
$$

with $t$ in days. We have

$$
\hat{c}_g = \frac{c_r + c_{o,\text{min}} e^p (n-1) + c_{fm} \hat{\alpha}_1 \left( e^{\hat{\alpha}_1 T_e} - 1 \right) + c_{im} \hat{\alpha}_1 \left( e^{\hat{\alpha}_1 T_s} - e^{\hat{\alpha}_1 T_w} \right)}{T_i}
$$

the following results are obtained:

$$
n^* = 7
$$

$$
T_i^* = 1877 \text{ days}
$$

$$
T_w^* = 1006 \text{ days}
$$

$$
c_g(n^*, T_i^*) = 0.1550 \text{ USD/day}
$$

So the system must be replaced every 5.15 years and must be overhauled every $1877/(7 - 1) = 313$ days. The warranty may be negotiated with the vendor from 2 years to $1006/365 = 2.76$ years.
some local minima may perturb seriously the finding of the
global minimum as it is observed in figure [6]. Optimization
is then done in terms of $T_i$.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$n$</th>
<th>$T_i$</th>
<th>$\hat{c}_p$</th>
<th>$T_w$</th>
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<tr>
<td>0.5</td>
<td>3</td>
<td>1391.2</td>
<td>0.1717</td>
<td>775.0</td>
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<tr>
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<td>5</td>
<td>1593.2</td>
<td>0.1654</td>
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<td>0.7</td>
<td>7</td>
<td>1877.8</td>
<td>0.1550</td>
<td>1005.6</td>
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<tr>
<td>0.8</td>
<td>10</td>
<td>2404.7</td>
<td>0.1395</td>
<td>1240.5</td>
</tr>
</tbody>
</table>

Table 1: Optimal solutions for different $p$

5. Conclusions
We have presented a model to relate the long term failure rate
with the improvement factor of the overhauls and their inter-
val. We established a cost optimization model to determine
optimal levels of preventive maintenance. The model includes
easily a negotiation criterion for extending the warranty pe-
period. The formulation has been simplified to permit the use of
standard spreadsheet solvers to solve the minimization prob-
lem. The extension of the model to considerate discounted
costs is straightforward.

In the near future the author is considering the relaxation
to the constraint that imposes equal intervals between over-
hauls. As failure rates increase, it would make sense to in-
crease the preventive strategy and further reduce costs.

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